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Colored Noise in Activated Rate Processes

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We consider bistable systems driven by stationary wideband Gaussian colored noise. We construct uniform asymptotic expansions of the stationary probability density function and of the activation rate, for small intensity ε and short correlation time τ of the noise. We find that for different values of the total power output ε/τ of the noise, different terms in the asymptotic expansions become dominant. For $\tau \ll \varepsilon$ we recover previously derived results, while for $\tau = O(\varepsilon)$ and $\varepsilon \ll \tau$ new results are obtained.

KEY WORDS: Colored noise; bistable dynamics; singular perturbation; first passage time; activated rate processes.

1. INTRODUCTION

The calculation of the activation rate of bistable dynamical systems driven by colored noise has been the focus of attention for a number of years (see refs. 1–15 and references therein). Even the simplest problem of one-dimensional dynamics forced by small additive Gaussian noise with very small correlation time led to unexpected difficulties. A number of theories have been proposed, leading to different and often conflicting results.(1–4,6,8–11,13,14) The ensuing confusion and controversy are due to the fact that the expansions obtained depend on two parameters, the noise intensity ε and the correlation time of the noise τ, in a nonuniform manner. The various analyses proposed were mostly based on the assumption that the expansion is regular in the small parameter τ, uniformly with respect to ε. Thus, first τ was used in a regular expansion for various non-Markovian Fokker–Planck-type equations (NMFPTE) for the probability density

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function (PDF) of the dynamics, and then the result was expanded for small ε. Kramers–Moyal-type expansions of the PDF were truncated after the second term to obtain a one-dimensional Fokker–Planck equation (FPE). Since the diffusion coefficients obtained this way were not always positive, they were replaced in an *ad hoc* manner by effective diffusion coefficients to ensure positivity. The various diffusion approximations to the non-Markovian dynamics were then used for the calculation of the activation rate and the mean first passage time (MFPT) from one stable state to the other.

The leading-order approximation for the MFPT for small τ is obviously the MFPT of the dynamics with the colored noise replaced by its small-τ limit, the white noise. On this point all theories agree and all reproduce the Smoluchowski–Kramers result.[17] The differences appear in the first asymptotic correction to this result. Such a correction expresses the influence of the finite correlation time on the activation rate or on the MFPT. The different approximations adopted for the calculation of the first asymptotic correction to the MFPT raise the following questions. First, is the approximation of the MFPT of a non-Markovian process by that of some approximating diffusion process correct up to second order in a small-τ asymptotic approximation? Further, is this approximation uniform with respect to other parameters of the problem, such as the intensity of the noise? Second, the calculation of the MFPT from a truncated master equation has been criticized in the literature,[24,26–28] since the tails of the PDF are in general not well approximated by the tails of the PDF of diffusion approximations obtained from such procedures, and the transition from one stable state to another is an event in the tail of the PDF. In view of this fact, is the resulting MFPT sufficiently well approximated by that obtained from the diffusion approximation, to second order in a small-τ expansion? Third, which of the different expansions for small τ is correct (see ref. 10 and references therein), and which, if any, is uniform with respect to the noise intensity ε?

Here we present a systematic theory, based on singular perturbation methods[20] and on clearly stated assumptions, for the calculation of the PDF and the MFPT as functions of the two small parameters ε and τ. We derive a uniform expansion for the PDF and the MFPT for all sufficiently small values of ε and τ. Our point of departure is the formulation of the problem as that of a two-dimensional Markov process whose components are the state variable and the driving noise. The PDF of the state variable is found as the marginal PDF of the two-dimensional PDF obtained from an asymptotic expansion of the solution of a two-dimensional Fokker–Planck equation. We construct this solution in two steps. First we consider the limit $\tau \ll 1$, followed by $\varepsilon \ll 1$, and then the reverse ordering. We show that both cases lead to the same expansion. Therefore the expansion is uniform. We calculate the MFPT as a solution of a partial differential equation[22–25] with both orderings of the parameters.

We note that there are at least three different situations to be considered, depending on the total power output ε/τ of the noise for τ, $\varepsilon \ll 1$: (i) $\varepsilon/\tau \ll 1$, (ii) $\varepsilon/\tau = O(1)$, and (iii) $\varepsilon/\tau \gg 1$. It is not *a priori* obvious that the same results will be valid in all three cases. We present results that are valid in all three cases and reduce to each other in the appropriate limits. We show that the expansion in each ordering of the limits is a rearrangement of the other, and we show that different terms in the expansion become dominant in different parameter ranges. Thus, we find that the expansions in refs. 1–15 are valid only in the restricted range $\tau \ll \varepsilon$, but not otherwise. We note that the $O(\tau)$ correction in the exponent of the expression for the MFPT in ref. 3 is incorrect, although a correction of this order of magnitude appears in the PDF. For other ranges of the parameters, we find additional terms that may dominate previously computed terms.

2. STATEMENT OF THE PROBLEM AND A SMALL-τ EXPANSION

One-dimensional bistable dynamics forced by small, wideband, colored noise $\xi(t)$ can be described by the stochastic differential equation

$$\dot{x}(t) = -U'(x(t)) + \xi(t) \tag{2.1}$$

where $U(x)$ is a bistable potential, e.g., $U(x) = x^4/4 - x^2/2$. Denoting the bandwidth of the noise $\xi(t)$ by α ($\alpha = 1/\tau$) and its intensity (spectral height) by ε, we can define $\xi(t)$ by the stochastic differential equation

$$\dot{\xi}(t) = -\alpha\xi(t) + (2\varepsilon)^{1/2}\,\alpha\dot{w}(t) \tag{2.2}$$

where $\dot{w}(t)$ is standard Gaussian white noise. The autocorrelation functions of $\dot{w}(t)$ and $\xi(t)$ are given by

$$\langle \dot{w}(t)\,\dot{w}(s)\rangle = \delta(t-s) \tag{2.3}$$

and

$$\langle \xi(t)\,\xi(s)\rangle = \varepsilon\alpha\exp(-\alpha\,|t-s|) \tag{2.4}$$

respectively. We note that the process $x(t)$ is not Markovian; however, the pair $(x(t), \xi(t))$ is Markovian. In the limit $\alpha \to \infty$, the colored noise $\xi(t)$ becomes white noise of intensity ε and (2.1) becomes the Smoluchowski equation describing the motion of an overdamped particle in a potential well. In this limit the process $x(t)$ becomes a Markovian diffusion process and its evolution can be described by a Fokker–Planck–Smoluchowski equation. In this case the calculation of the MFPT over the potential barrier follows the standard methods for Markovian diffusions.[21–24] The result is the well-known Kramers formula[17]

$$T_{\infty} = \frac{\pi}{\omega_A \omega_C} e^{\Delta U/\varepsilon} \tag{2.5}$$

where ΔU is the height of the potential barrier, and ω_A and ω_C are the frequencies of vibration at the bottom and at the top of the potential well, respectively. Note that T_{∞} in (2.5) is the MFPT to the top of the barrier, while the mean time to cross over is twice that number, since trajectories that reach the top cross over or return with equal probabilities.[25]

Defining $y = -U'(x) + \xi(t)$, we rewrite the system (2.1), (2.2) in the form

$$\dot{x}(t) = y(t) \tag{2.6}$$

$$\dot{y}(t) = -[U''(x) + \alpha]\, y(t) - \alpha U'(x) + (2\varepsilon)^{1/2}\, \alpha \dot{w}(t) \tag{2.7}$$

Now the pair $(x(t), y(t))$ is a two-dimensional diffusion process whose joint PDF satisfies the FPE

$$p_t = -y p_x + \{[(U''(x) + \alpha)\, y + \alpha U'(x)]\, p\}_y + \varepsilon \alpha^2 p_{yy} \tag{2.8}$$

The two-dimensional dynamical system (2.6), (2.7) has, in the absence of noise [i.e., for $\varepsilon = 0$ in (2.7)], a stable equilibrium point (an attractor) at

$$x = x_A, \qquad y = 0 \tag{2.9}$$

and an unstable equilibrium point (a saddle point) at

$$x = x_C, \qquad y = 0 \tag{2.10}$$

(see Fig. 1). The domain of attraction D of the stable equilibrium point is bounded by a separatrix Γ. The curve Γ in the (x, y) plane consists of the two tajectories of (2.6), (2.7) (with $\varepsilon = 0$) that converge to the saddle point.

The MFPT for the system (2.1), (2.2) is the mean time for the pair $(x(t), y(t))$ to hit Γ for the first time. Note that the MFPT is *not* the mean time for $x(t)$ to hit x_C for the first time, since once the trajectory of (2.6), (2.7) crosses Γ, it drifts deterministically across the line $x = x_C$ in a relatively short time [independent of the noise $\dot{w}(t)$].

Next we discuss the various methods proposed for the calculation of the MFPT. In ref. 3 (and refs. 4, 5, 14, and 15 therein) and ref. 10 (and ref. 5 therein) a Fokker–Planck-like equation for the PDF of $x(t)$ is constructed. For large α, various procedures for the derivation of an approximate Fokker–Planck equation for the PDF are adopted.[11,13] The resulting FPE has the form

$$p_t = [U'(x)\,p]_x + [D(x, t)\,p]_{xx} \tag{2.11}$$

where $D(x, t)$ is a state-dependent diffusion coefficient. Various expressions for the stationary coefficient $D(x) = \lim_{t \to \infty} D(x, t)$ were given.[10,13] The MFPT for the system (2.1), (2.2) is identified with that of the one-

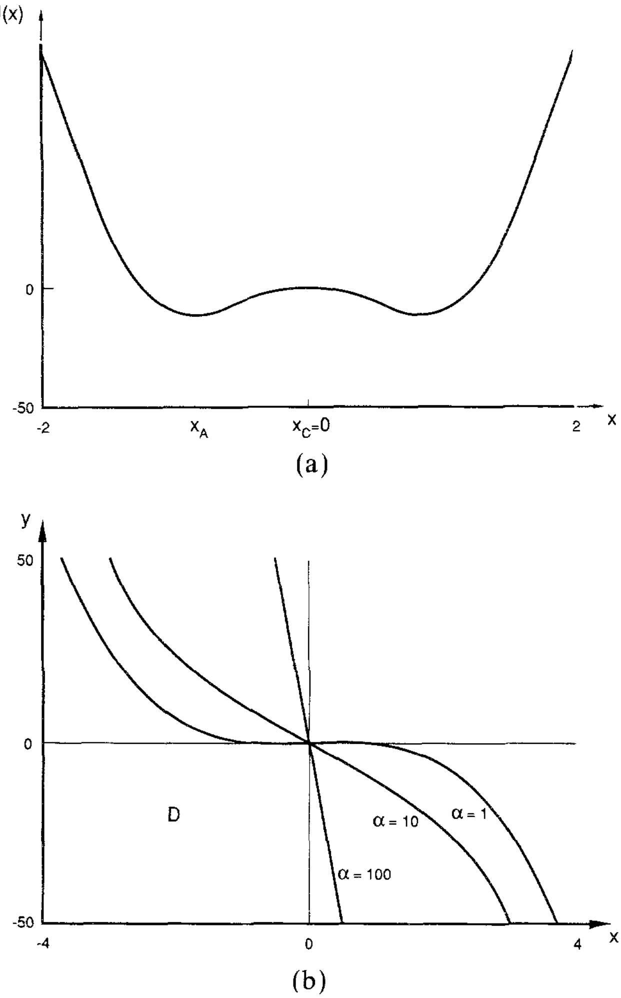

Fig. 1. (a) The potential $U(x) = x^4/4 - x^2/2$. (b) Separatrices for the potential $x^4/4 - x^2/2$ for $\alpha = 1$, 10, and 100, respectively.

dimensional state-dependent diffusion process described by (2.11). Such a diffusion process is the solution of the Itô stochastic equation[21]

$$\dot{x}(t) = -U'(x) + [2D(x)]^{1/2}\,\dot{w}(t) \tag{2.12}$$

This identification is not obvious and certainly requires some clarification. It should be noted that passage out of the domain of attraction D may occur on one hand prior to hitting the line $x = x_C$, and on the other hand a transition over this line may result in an immediate return into D if at the time of such a transition the value of $\xi(t)$ is large and negative.

Next we examine this procedure by constructing an asymptotic expansion of the stationary PDF of $x(t)$ in the limit $\alpha \to \infty$ directly from the stationary two-dimensional FPE (2.8). First we change variables in (2.8) by setting

$$y = \alpha^{1/2} z \tag{2.13}$$

and rewrite the stationary FPE in the form

$$\begin{aligned}\alpha[\varepsilon p_{zz} + (zp)_z] + \alpha^{1/2}[-zp_x + U'(x)\,p_z] + U''(x)(zp)_z \\ \equiv \alpha L_0\,p + \alpha^{1/2} L_1\,p + L_2\,p = 0\end{aligned} \tag{2.14}$$

The joint stationary PDF of (x, y) in the limit $\alpha \to \infty$ can be easily obtained by expanding the solution of the stationary FPE (2.14) in an asymptotic series in powers of $\alpha^{-1/2}$. The leading term and the first correction in this expansion are given by[30]

$$p \sim C\left[1 + U''(x)\left(\frac{3}{2\alpha} - \frac{z^2}{2\varepsilon}\right) - \frac{[U'(x)]^2}{2\varepsilon\alpha}\right]\exp\left[-\frac{[U(x) + z^2/2]}{\varepsilon}\right] \tag{2.15}$$

where C is a normalization constant. A one-dimensional diffusion approximation to the non-Markovian dynamics $x(t)$ for $1/\alpha \ll 1$ (actually, for $1/\alpha \ll \varepsilon$) is obtained as follows. First, the marginal stationary PDF of $x(t)$ is obtained from (2.15) by integration with respect to z as

$$p(x) \sim D(x)\exp[-U(x)/\varepsilon] \tag{2.16}$$

where

$$D(x) = 1 + \frac{1}{\alpha}\left[U''(x) - \frac{[U'(x)]^2}{2\varepsilon}\right] \tag{2.17}$$

Next, the approximating diffusion process, whose stationary PDF is given by (2.16), is defined by the Itô stochastic differential equation

$$\dot{x}(t) = m(x(t)) + [2\varepsilon\sigma(x(t))]^{1/2}\,\dot{w}(t) \tag{2.18}$$

with

$$m(x) = -U'(x)/D(x) \tag{2.19}$$

and

$$\sigma(x) = 1/D(x) \tag{2.20}$$

Then, using standard methods,[17-19] the MFPT T of the process defined in (2.18) is found as

$$T = \Omega \exp(\Delta U/\varepsilon) \tag{2.21}$$

where the attempt frequency Ω is given by

$$\Omega = \pi \left[\frac{\sigma(x_C)}{-m'(x_C)\, m'(x_A)\, \sigma(x_A)} \right]^{1/2} \tag{2.22}$$

For the potential $U(x) = x^4/4 - x^2/2$, (2.21) and (2.22) give

$$T \sim T_\infty [1 + 3/(2\alpha) + \cdots] \tag{2.23}$$

which agrees with refs. 4, 9b, 10, and 11.

The validity of such a procedure for calculating the MFPT for the non-Markovian process $x(t)$ is questionable, in view of the criticism of the diffusion approximation to the master equation.[24,26-28] In the next section we calculate the MFPT of the two-dimensional Markov process $(x(t), y(t))$ and show that (2.23) is a valid approximation if $\tau = 1/\alpha \ll \varepsilon \ll 1$.

The large-α expansion for the MFPT of the two-dimensional process (x, y) to the separatrix Γ is obtained as follows. We denote the MFPT, given the initial state (x, y), by $T(x, y)$ and recall[21] that it satisfies the boundary value problem

$$\varepsilon\alpha^2 T_{yy} - \{[\alpha + U''(x)]\, y + \alpha U'(x)\}\, T_y + y T_x = -1 \qquad \text{in} \quad D \tag{2.24}$$

and

$$T(x, y) = 0 \qquad \text{on} \quad \Gamma \tag{2.25}$$

Then we construct an asymptotic expansion of $T(x, y)$ in powers of $1/\alpha$.[30] The MFPT is $T(x_A, 0)$, which is given by

$$T(x_A, 0) = T_\infty \left[1 + \frac{1}{2\alpha} (\omega_A^2 + \omega_C^2) + O\left(\frac{1}{\alpha^2}\right) \right] \tag{2.26}$$

where T_∞ is given in (2.5). It follows that the expression for the MFPT containing the first-order correction to the white noise result for the potential

$$U(x) = x^4/4 - x^2/2 \tag{2.27}$$

is given by

$$T \sim T_0(x_A)[1 + 3/(2\alpha)] \tag{2.28}$$

Formula (2.28) is valid for $\tau = 1/\alpha \ll \varepsilon \ll 1$. However, this formula is not uniformly valid for all ε. Indeed, for $\varepsilon \ll 1/\alpha = \tau \ll 1$, other terms enter the expansion, and in fact dominate the $O(1/\alpha)$ term above.[12,16]

3. A UNIFORM EXPANSION OF THE MFPT

The asymptotic expansion of the MFPT for small ε was given in refs. 16 and 30. Following ref. 30, we present two expansions of T. First we construct an expansion by considering the limits $\varepsilon \ll 1/\alpha \ll 1$, and then we consider the limits $1/\alpha \ll \varepsilon \ll 1$. We show that the first expansion agrees with the second; thus, the expansion is uniform for $\varepsilon \ll 1$ and $\alpha \gg 1$.

In the first case ($\varepsilon \ll 1/\alpha \ll 1$), we construct an asymptotic solution to the stationary FPE (2.8) in the WKB form

$$p = K(x, y, \varepsilon, \alpha) \exp[-\phi(x, y, \alpha)/\varepsilon] \tag{3.1}$$

where $K(x, y, \varepsilon, \alpha) \sim K^0(x, y, \alpha) + \varepsilon K'(x, y, \alpha) + \cdots$. Inserting (3.1) into (2.8), we obtain the eikonal equation

$$y\phi_x - [(U'' + \alpha) y + \alpha U'] \phi_y + \alpha^2 \phi_y^2 = 0 \tag{3.2}$$

The leading term K^0 in the expansion of the preexponential factor K with respect to ε satisfies the transport equation

$$yK_x^0 - \{[y(\alpha + U''(x)) + \alpha U'(x)] K^0\}_y + \alpha^2[2K_y^0 \phi_y + K^0 \phi_{yy}] = 0 \tag{3.3}$$

For large α we expand the solution of (3.2) in powers of $1/\alpha$. We note that such an expansion is assumed only for values of x and y that are $O(1)$ relative to the expansion parameter α. Thus, we assume

$$\phi \sim \phi_0 + \phi_1/\alpha + \phi_2/\alpha_2 + \cdots \tag{3.4}$$

We obtain

$$\phi_0 = U(x) \tag{3.5}$$

$$\phi_1 = \frac{y^2}{2} + \frac{[U'(x)]^2}{2} \tag{3.6}$$

$$\phi_2 = \frac{y^2 U''(x)}{2} - \frac{1}{2} \int_{x_A}^{x} [U'(s)]^2 U'''(s)\, ds \tag{3.7}$$

$$\phi_3 = -U'''(x) \left\{ \frac{y^3}{6} + \frac{U'(x)\, y^2}{4} - \frac{5}{12} [U'(x)]^3 \right\} \tag{3.8}$$

$$\begin{aligned} \phi_4 = {} & \frac{y^4}{24} U^{\mathrm{IV}}(x) + \frac{y^3}{12} \left\{ 3U''(x)\, U'''(x) + \frac{5}{8} U'(x)\, U^{\mathrm{IV}}(x) \right\} \\ & + \frac{5y^2}{24} \{3U'(x)\, U''(x)\, U'''(x) + [U'(x)]^2\, U^{\mathrm{IV}}(x)\} \\ & - \int_{x_A}^{x_C} (U'(x))^3\, (U'''(x))^2\, dx \end{aligned} \tag{3.9}$$

and so on. Similarly, we expand the solution of (3.3) as

$$K \sim K_0 + \frac{K_1}{\alpha} + \frac{K_2}{\alpha^2} + \cdots$$

and obtain

$$K_0^0 = 1 \tag{3.10}$$

$$K_1^0 = \frac{3U''(x)}{2} \tag{3.11}$$

$$K_2^0 = -\frac{y}{2} U'''(x) - \frac{5}{4} U'(x)\, U'''(x) + \frac{5}{8} [U''(x)]^2 \tag{3.12}$$

and so on.

To find the MFPT we recall the formula[29]

$$T = \pi \left| \frac{H_C}{H_A} \right|^{1/2} \frac{K_A e^{(\phi_C - \phi_A)/\varepsilon}}{K_C \lambda_p} (1 + O(\varepsilon)) \tag{3.13}$$

where H_i are the Hessians of ϕ at x_i, λ_p is the unique positive eigenvalue of the linearized system (2.6), (2.7) about the saddle point, and K_i^0 and ϕ_i are the values of K^0 and ϕ at the point $(x_i, 0)$ $(i = A, C)$. Using (3.1) with (3.4)–(3.12) in (3.13), we find

$$T \sim \frac{\pi}{\omega_A \omega_C} \frac{(1 - 2\omega_C^2/\alpha + \omega_C^4/\alpha^2)^{1/2}}{(1 + 2\omega_A^2/\alpha + \omega_A^4/\alpha^2)^{1/2}} \frac{1 + 3\omega_A^2/2\alpha + 5\omega_A^4/8\alpha^2 + O(\varepsilon)}{1 - 3\omega_C^2/2\alpha + 5\omega_C^4/8\alpha^2 + O(\varepsilon)} \times \exp\left\{\frac{1}{\varepsilon} \Delta\phi\right\}$$

$$\Delta\phi = \left[U(x_C) - U(x_A) - \frac{1}{2\alpha^2} \int_{x_A}^{x_C} (U')^2\, U'''\, dx + \frac{1}{\alpha^4} \int_{x_A}^{x_C} (U')^3\, (U''')^2\, d\bar{x} \right] \tag{3.14}$$

We note that (3.14) agrees with the expansion of (2.21)–(2.22) for large α, and also with the expansion (2.26). We observe that the first two terms in $\Delta\phi$ were also obtained in refs. 31, 32, and all three terms in $\Delta\phi$ were obtained in ref. 33. However, the preexponential factor in (3.14) was not obtained.

Next we consider the second case, $1/\alpha \ll \varepsilon \ll 1$. We construct a WKB solution to (2.8) in the form

$$p \sim \exp\left[-\alpha \left(\psi_0 + \frac{1}{\alpha} \psi_1 + \cdots \right) \right] \tag{3.15}$$

where ψ_j are functions of the variables x and $v = x - y/\alpha$. We find that

$$\psi_0 = \frac{(x-v)^2}{2\varepsilon} \tag{3.16}$$

$$\psi_1 = \frac{1}{\varepsilon}\left[U'(x)(x-v) + U(v)\right] \tag{3.17}$$

$$\psi_2 = \frac{1}{2\varepsilon}\left[U'(x) - U'(v)\right]^2 - \int_v^x F(t, v)\, dt$$

$$+ \frac{1}{\varepsilon} U'(v) \int_v^x G(t, v)\, dt + \frac{1}{2\varepsilon}\left[U'(v)\right]^2 - \frac{3}{2} U''(v) \tag{3.18}$$

where

$$F(t, v) = \frac{U''(t) - U''(v)}{t - v} \tag{3.19}$$

and

$$G(t, v) = \frac{U'(t) - U'(v)}{t - v} \tag{3.20}$$

Finally,

$$\psi_3 = -\int_v^x U''(t)\, \psi_{2,v}(t, v)\, dt - \int_v^x \frac{B(t, v)}{t - v}\, dt + K(v) \tag{3.21}$$

where

$$B(t, v) = U'(t)\, \psi_{2,v}(t, v) - \varepsilon \psi_{2,vv}(t, v) + 2\varepsilon \psi_{1,v}(t, v)\, \psi_{2,v}(t, v) \tag{3.22}$$

and

$$K(v) = \frac{K_0(v)}{\varepsilon} + K_1(v) + \varepsilon K_2(v) \tag{3.23}$$

with

$$K_0(v) = -\frac{1}{2} \int_{x_A}^v \left[U'(t)\right]^2 U'''(t)\, dt \tag{3.24}$$

$$K_1(v) = \frac{5}{4} U'(v)\, U'''(v) + \frac{1}{2}\left[U''(v)\right]^2 \tag{3.25}$$

and

$$K_2(v) = -\frac{5}{8} U^{\text{iv}}(v) \tag{3.26}$$

The expansions (3.1) and (3.15) can be compared by

$$\left[1 + \frac{1}{\alpha} K_1 + \frac{1}{\alpha^2} K_2 + O\left(\frac{1}{\alpha^3}\right)\right] \exp\left\{-\frac{1}{\varepsilon}\left[\phi_0 + \frac{1}{\alpha}\phi_1 + \frac{1}{\alpha^2}\phi_2 + O\left(\frac{1}{\alpha^3}\right)\right]\right\}$$

$$= \exp\left\{-\left[\alpha\psi_0 + \psi_1 + \frac{1}{\alpha}\psi_2 + \frac{1}{\alpha^2}\psi_3 + O\left(\frac{1}{\alpha^3}\right)\right]\right\} \tag{3.27}$$

In (3.27) terms up to orders $O(1/\varepsilon\alpha^2)$ and $O(1/\alpha^2)$ have been kept. Finally, to calculate the MFPT, we use (3.15)–(3.26) in (3.13). We obtain, as in (2.21),

$$T = \Omega e^{\Delta U/\varepsilon} \tag{3.28}$$

with

$$\Omega = \frac{\pi}{\omega_A \omega_C} \frac{\{1 + (1/\alpha)[-2\omega_C^2 + O(\varepsilon)] + O(1/\alpha^2)\}^{1/2}}{\{1 - (1/\alpha)[-2\omega_A^2 + O(\varepsilon)] + O(1/\alpha^2)\}^{1/2}} \Omega_0(\alpha, \varepsilon) \tag{3.29}$$

where

$$\Omega_0(\alpha, \varepsilon) = \exp\left\{ -\frac{1}{2\varepsilon\alpha^2} \int_{x_A}^{x_C} [U'(t)]^2 U'''(t)\, dt + \frac{3}{2\alpha} (\omega_A^2 + \omega_C^2) + \frac{1}{2\alpha^2} (\omega_C^4 - \omega_A^4) + O\left(\frac{1}{\alpha^3}, \frac{\varepsilon}{\alpha^2}\right)\right\} \tag{3.30}$$

For $1/\alpha \ll \varepsilon$, (3.28)–(3.30) reduce to (2.26), while for $1/\alpha \gg \varepsilon$, (3.28)–(3.30) reduce to (3.14). We observe that in the latter case, the term $O(1/\varepsilon\alpha^2)$ in (3.30) may in fact dominate the term which is $O(1/\alpha)$. In a similar manner (3.14) considered for $1/\alpha \ll \varepsilon$, can be shown to reduce to (3.28)–(3.30). Thus each expansion can be obtained from the other by appropriately rearranging terms. Thus, (3.28)–(3.30) and (3.14) are uniform expansions of the MFPT for all $\varepsilon \ll 1$ and $\alpha \gg 1$, while (2.21)–(2.22) and (2.26), which were obtained in refs. 4, 9b, 10, and 11, are not uniform throughout this range, but rather are only valid for $1/\alpha \ll \varepsilon \ll 1$. The results (2.21)–(2.22) and (2.26) diverge significantly from our uniform results (3.14) and (3.28) when $1/\alpha$ exceeds ε, and this divergence increases with $1/\alpha$. This prediction is confirmed by recent numerical simulations.[34] The dependence of the MFPT on $1/\varepsilon\alpha^2$ is exponential and in general cannot be expressed as a preexponential factor. If $\varepsilon = O(1/\alpha)$, then $O(1/\alpha)$ and $O(1/\varepsilon\alpha^2)$ terms are comparable and both must be retained as preexponential factors in (3.28)–(3.30). If, however, $O(1/\alpha^2) \ll \varepsilon \ll O(1/\alpha)$, the $O(1/\varepsilon\alpha^2)$ term dominates the $O(1/\alpha)$ term, though it can still be expressed as a preexponential factor. Finally, if $\varepsilon = O(1/\alpha^2)$, the $O(1/\varepsilon\alpha^2)$ term dominates all other corrections to the Kramers–Smoluchowski formula (2.5). It must be retained in the exponent, and cannot be expressed as a preexponential factor.

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